Midsemestral examination
M.Math. Ist year
Subject - Number Theory : Instructor - B.Sury
October 16, 2003

Not all questions carry equal marks. Any score of more than 100 will be counted as 100. Attempt at the most ONE question out of questions 1 to 3, TWO out of questions 4 to 6, THREE out of questions 7 to 10, and THREE out of questions 11 to 15. Question 16 is compulsory.

Q 1 (12 marks).
Use the quadratic reciprocity law to decide whether $x^2 + 5x \equiv 12$ *mod* 31 *has solutions. If it has a solution, find one.*

Q 2 (8 marks).
(a) *If* $p \equiv 1$ *mod* 4, *prove that* $\sum_{a=1}^{p-1} a(\frac{a}{p}) = 0$.
(b) *If* $p > 3$, *prove that* $\sum_{(\frac{a}{p})=1} a \equiv 0$ *mod* p.

Q 3 (10 marks).
Let $m_1, \cdots, m_r$ *be pairwise coprime natural numbers. Let* $a_1, \cdots, a_r$ *be so that* $(a_i, m_i) = 1$ *for all* i. *Prove that there are infinitely many prime numbers* p *such that* $p \equiv a_i$ *mod* m_i *for all* $i \leq r$.
Hint : You may use Dirichlet's theorem.

Q 4 (12 marks).
Show that every element of a finite field is a sum of two squares.

Q 5 (15 marks).
Let p *be an odd prime and* $(a_i, b_i) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}; 1 \leq i \leq 3p-2$.
(a) *Apply the Chevalley-Warning theorem to the set of three polynomials*
$\sum_{i=1}^{3p-2} a_i X_i^{p-1}, \sum_{i=1}^{3p-2} b_i X_i^{p-1}$ *and* $\sum_{i=1}^{3p-2} X_i^{p-1}$
to obtain a set T *of cardinality* p *or* $2p$ *for which* $\sum_{i \in T}(a_i, b_i) = (0,0) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$.
(b) *If* T *has cardinality* $2p$, *say,* $T = \{1, 2, \cdots, 2p\}$, *apply the Chevalley-Warning theorem to the set of polynomials*
$\sum_{i=1}^{2p-1} a_i X_i^{p-1}, \sum_{i=1}^{2p-1} b_i X_i^{p-1}$, *and* $\sum_{i=2p}^{3p-2} X_i^{p-1}$
to obtain a set S *with at the most* p *elements so that* $\sum_{i \in S}(a_i, b_i) = (0,0) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$.

Q 6 (12 marks).

Let F be a finite field of cardinality q and let $v : F \to \mathbf{Z}$ be the map defined by $v(0) = q - 1$ and $v(a) = -1$ for $a \neq 0$.
(a) *If $b \in F$, prove for each m that*

$$\sum_{c_1+\cdots+c_m=b} v(c_1)\cdots v(c_m) = v(b)q^{m-1}.$$

(b) *Using (a) or otherwise, show that when q is even, the number of solutions of $x_1x_2 + x_3x_4 + x_{99}x_{100} = 1$ in F^{100} is $q^{99} - q^{49}$.*

Q 7 (10 marks).
Let $f = a_0 + a_1X + \cdots + a_nX^n \in \mathbf{Q}_p[X]$ be irreducible of degree $n \geq 1$, where $f(0) \neq 0$. Use Hensel's lemma to prove that if $a_0, a_n \in \mathbf{Z}_p$, then $f \in \mathbf{Z}_p[X]$.

Q 8 (6 marks).
Show that $\mathbf{Z}_p$ is a Hausdorff space in which connected sets are points.

Q 9 (12 marks).
Let p be an odd prime and a, b, c integers such that p does not divide any of them. Use Hensel's lemma (in several variables) to prove that $aX^2 + bY^2 + cZ^2 = 0$ has a non-trivial solution in $\mathbf{Q}_p$.
Hint : You may use the Chevalley-Warning theorem for the above polynomial reduced modulo p.

Q 10 (8 marks).
Define $\mathbf{Z}_p$. Show that for each $x \in \mathbf{Z}_p$, there is a unique sequence $\{x_n\}$ of integers with $0 \leq x_n < p^n$ and $x_{n+1} \equiv x_n$ mod p^n for all n, which converges to x in $\mathbf{Q}_p$.

Q 11 (6 marks).
If the Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ converges at $s = s_0$, prove that it converges absolutely for all s with Re $s >$ Re $s_0 + 1$.
Give an example (without proof) to show that the abscissa of absolute convergence σ_a could be equal to $\sigma_c + 1$, where σ_c is the abscissa of convergence.

Q 12 (12 marks).
If $f(s) = \sum_{n\geq 1} a_n n^{-s} = \zeta(s)\zeta(s-2)$, prove that $f(s)$ converges for Re $s > 3$. Find an expression for $a(n)$. What is $a(10)$?

Q 13 (8 marks).
Prove, using Abel's partial summation or otherwise, that the Riemann zeta

function $\zeta(s) = \sum_{n\geq 1} n^{-s}$ *has a meromorphic continuation to Re* $s > 0$ *with a simple pole at* $s = 1$ *and no other poles.*

Q 14 (10 marks).
Let $f \in \mathbf{Z}[X]$ *be of degree* $n \geq 1$*. If, for each prime* p*,* $f(p) = q^r$ *for some prime* q *and some* $r \geq 1$*, prove that* $f(X) = X^n$*.*

Q 15 (12 marks).
Given that $\zeta(s) - \frac{1}{s-1}$ *is analytic in Re* $s > 0$*, and that* $L(1, \chi) \neq 0$ *for each nontrivial Dirichlet character* χ *mod* n*, deduce that the set* P *of primes* $\equiv -1$ *mod* n *satisfies* $\frac{\sum_{p\in P} p^{-s}}{log \frac{1}{s-1}} \to 1$ *as* $s \to 1^+$*.*

Q 16 (12 marks).
Are these true or false - indicate one or two lines of reasoning:
(i) $\mathbf{Z}_p^* \cup \{0\} = \mathbf{Z}_p$.
(ii) *If* $a := \sum_{i\geq 0} a_i p^i \in \mathbf{Z}_p$ *with* $a_i = a_{10+i}$ *for all* i*, then* $a \in \mathbf{Q}$.
(iii) $X^p - p$ *has a root in* $\mathbf{Q}_p$*.*
(iv) $\sum_{a=1}^{p-1} (\frac{a}{p}) = 0$*.*
(v) $\mathbf{Z}/p^2\mathbf{Z} \cong \mathbf{F}_{p^2}$ *as rings.*
(vi) If $\sum_{n\geq 1} \frac{1}{n^s(n+1)}$ *has abscissa of convergence* ρ*, then* $\sum_{n\geq 1} \frac{1}{n^\rho(n+1)}$ *converges.*